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## Autohomeomorphism Groups of 0-dimensional Spaces

by

J. de Groot and R. H. McDowell<sup>1)</sup>

If  $T$  is a topological space, we denote by  $A(T)$  the group of all homeomorphisms of  $T$  onto itself. In [2], it was shown that given an arbitrary group  $G$ , one can find a topological space  $T$  such that  $G$  and  $A(T)$  are isomorphic; in fact, such a  $T$  can be found among the compact connected Hausdorff spaces. In general, no such  $T$  can be found among the spaces with a base of open – and – closed sets, i.e., the spaces  $T$  such that  $\dim T = 0$ . The present paper investigates the following question. What can be said, in general, about  $A(T)$  if  $T$  is a completely regular Hausdorff space and  $\dim T = 0$ ?

If  $\alpha$  is any cardinal  $\geq 1$ , we shall denote by  $S_\alpha$  the restricted permutation group on  $\alpha$  objects; that is, the group of all those permutations which involve only finitely many objects. We will find it convenient to let  $S_0$  denote the group of one element.  $\Sigma C_2$  will denote the direct sum of  $\aleph_1$  groups of order two. Throughout this paper, “space” will be used to mean “completely regular Hausdorff space”. For any 0-dimensional space  $T$ , we shall show that  $A(T)$  must

(1) consist of a single element (in which case we say  $T$  is “rigid”),  
(2) contain a subgroup  $S_\alpha$  for some  $\alpha$ ,  
or (3) contain a subgroup of the form  $S_\alpha + \Sigma C_2$ . This result is best possible, in the sense that for any cardinal  $\alpha$ , we can construct spaces whose autohomeomorphism group is precisely  $S_\alpha$  or  $S_\alpha + \Sigma C_2$ . We produce examples of arbitrarily high weight,<sup>2)</sup> but we leave open the problem of constructing *compact rigid* 0-dimensional spaces of arbitrarily high weight.

In particular, if  $T$  is dense in itself,  $A(T)$  equals the unit

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<sup>2)</sup> The weight of a space is  $m$  if there exists an open base of  $m$  and not less than  $m$  sets.

element or contains a subgroup  $\Sigma C_2$ . On the other hand, one can construct compact 0-dimensional Hausdorff spaces  $H$ , dense in itself, for which  $A(H) = 1$  or  $A(H)$  equals the direct sum of continuously many groups of order two (in the last case one takes the Čech-Stone compactification of [2; § 5, example I]).

Some of the results of this paper were announced in [3].

### I. $A(T)$ for 0-dimensional Spaces

**1.1. LEMMA.** Let  $\{x_i\}$  and  $\{y_i\}$ ,  $i \in N$  (the natural numbers) be sets of distinct isolated points in the space  $T$  such that, for every  $J \subset N$ ,  $\{x_j\}$  and  $\{y_j\}$ ,  $j \in J$ , have identical boundaries in  $T$ ; then  $T$  admits of uncountably many distinct autohomeomorphisms of order two.

**PROOF.** It is easy to see that the map interchanging  $x_i$  and  $y_i$  for each  $i$  in  $N$ , and leaving all other points of  $T$  fixed, is an autohomeomorphism; the same is clearly true for every subset  $J$  of  $N$ , and there are uncountably many such subsets.

In what follows, we shall need the following well known (and easily proved) result from group theory.

**1.2. PROPOSITION.** If  $G$  is a group in which all elements distinct from the identity have order two, then  $G$  can be represented as the direct sum of cyclic groups of order two.

**1.3. THEOREM.** Let  $T$  be a 0-dimensional completely regular Hausdorff space, containing  $\alpha$  isolated points ( $\alpha$  may be 0). Then either  $A(T) = S_\alpha$ , or  $A(T)$  contains a subgroup of the form  $S_\alpha + \Sigma C_2$ .

**PROOF.**  $A(T)$  clearly contains a subgroup isomorphic to  $S_\alpha$ , since every one - one onto map moving a finite number of isolated points, and leaving all other points fixed, is a homeomorphism. Thus we need only show that if  $T$  admits any autohomeomorphism which does *more* than this, then  $T$  contains a subgroup isomorphic to  $S_\alpha + \Sigma C_2$ .

Note first of all that if  $\alpha > \aleph_0$ , there is no problem, since  $S_\alpha$  itself contains such a subgroup. So we assume  $\alpha \leq \aleph_0$ , and we distinguish two cases.

(1) There is an autohomeomorphism  $\varphi$  on  $T$  which moves a non-isolated point  $p$ . Then we can find an open - and - closed set  $U$  containing  $p$  such that  $U \cap \varphi(U) = \emptyset$ . If  $U$  has no countable base, we can find more than  $\aleph_0$  distinct open- and -closed subsets  $K \subset U$ , and interchanging  $K$  and  $\varphi(K)$  gives us an autohomeo-

morphism of order two. If  $U$  has a countable base, let  $D = \{x_i\}$  be the set of all isolated points in  $U$ . If  $D$  is finite, then  $M = (U \setminus D) \cup \varphi(U \setminus D)$  is open-and-closed, dense in itself, separable, metrizable and 0-dimensional, and is therefore homeomorphic to a dense-in-itself subset of the Cantor set. Since  $M$  is not rigid,  $A(M)$  (and hence  $A(T)$ ) contains a subgroup of the form  $\Sigma C_2$ , by [2; p. 90, (i)]. If  $D$  is infinite and closed, let  $\{x_i\}$  be any enumeration of  $D$ ; then  $\{x_i\}$  and  $\{\varphi(x_i)\}$  satisfy the hypotheses of Lemma 1.1; if  $D$  is not closed, it has a limit point  $q$  and a subsequence  $\{y_i\}$  converging to  $q$ . In that case,  $\{y_{2i-1}\}$  and  $\{y_{2i}\}$  satisfy the hypotheses of 1.1.

(2) No autohomeomorphism moves a non-isolated point. Let  $\varphi$  be a homeomorphism moving an infinite set of isolated points  $\{y_i\}$ . If we can find a set of isolated points  $\{x_i\}$  such that  $\{x_i\} \cap \{\varphi x_i\} = \emptyset$ , then  $\{x_i\}$  and  $\{\varphi x_i\}$  clearly satisfy the hypotheses of 1.1. But such a set  $\{x_i\}$  is easily found, for if there is a  $y \in \{y_i\}$  with infinite orbit, let  $x_i = \varphi^{2i}y$ ; if each  $y_i$  has finite orbit, form  $\{x_i\}$  by choosing one point from each of the orbits determined by  $y_i$ .

It follows that  $A(T)$  contains a group isomorphic to  $\Sigma C_2$ ; from the construction, it is easily seen that by dividing the isolated points into two disjoint infinite sets if necessary, one can find a subgroup isomorphic to  $S_\alpha + \Sigma C_2$ .

It should be pointed out that in only one case in the proof of 1.3 do we fail to find *continuously* many distinct autohomeomorphisms of order two. We could replace  $\Sigma C_2$  in the statement of the theorem by the direct sum of continuously many groups of order two if we could prove the following: if  $U$  and  $V$  are 0-dimensional, disjoint homeomorphic spaces having no countable base, and  $X = U \cup V$ , then  $A(X)$  contains  $\mathfrak{c}$  elements of order two.

## II. Rigid Spaces

In this section, we extend the methods of [2] to produce rigid 0-dimensional spaces of arbitrary (infinite) weight. We shall require some ideas in the theory of uniform spaces; the reader is referred to [1] and [4] for a development of these ideas.

First, we extend a metric space theorem to uniform spaces in a routine manner.

**2.1. DEFINITION.** An intersection of  $\mathfrak{m}$  open sets will be called a  $G_{\mathfrak{m}\delta}$ -set; a  $G_{\aleph_0\delta}$ -set will be called, as usual, a  $G_\delta$ -set.

**2.2. THEOREM.** Let  $X$  be a completely regular Hausdorff space

of weight  $\mathbf{m}$ , complete in a uniformity  $\mathcal{D}$  generated by a set  $D$  of  $\mathbf{m}$  pseudometrics. Then every continuous map  $f$  from a subset  $H$  of  $X$  into  $X$  can be extended continuously to a map  $\tilde{f}$  from a  $G_{\mathbf{m}\delta}$ -set  $G \supset H$  into  $X$ .

PROOF. For each  $d \in \mathcal{D}$ , and each  $x \in \bar{H}$ , let  $\omega_d(x)$  be the oscillation of  $f$  at  $x$  with respect to  $d$ . Let

$$G_d = \{x \in \bar{H} : \omega_d(x) = 0\}.$$

$G_d$  is evidently a  $G_\delta$ -set. Let

$$G = \bigcap_{d \in \mathcal{D}} G_d;$$

then  $G \supset H$  is a  $G_{\mathbf{m}\delta}$ -set.

Now  $f$  can be extended continuously over  $G$ . For let  $\{h_\alpha\}$  be any net in  $H$  converging to a point  $x \in G$ . Then, in the uniformity generated by  $\mathcal{D}$ ,  $\{f(h_\alpha)\}$  is a Cauchy net, by the definition of  $G$ . Hence  $\{f(h_\alpha)\}$  converges to some point  $p \in X$ ; set  $\tilde{f}(x) = p$ .  $\tilde{f}$  is evidently continuous at  $x$ .

Now, using 2.2, we extend some of the results in [2].

2.3. DEFINITION. If  $X$  is a topological space, and  $f$  a map from a subset of  $X$  into  $X$ , then  $f$  is called a *continuous displacement of order  $\mathbf{M}$*  if  $f$  is continuous, and is a displacement of order  $\mathbf{M}$ . A continuous displacement of order  $\mathbf{c}$  will be called, as usual, a continuous displacement [2; § 2].

2.4. THEOREM. Let  $X$  be a completely regular Hausdorff space of weight  $\mathbf{m}$ , complete in a uniformity  $\mathcal{D}$  generated by  $\mathbf{m}$  pseudometrics, and let  $|X| = 2^{\mathbf{m}} = \mathbf{M}$ . Further, let  $\{K_\beta\}$  be any family of  $\mathbf{M}$  subsets of  $X$ , each of cardinal  $\mathbf{M}$ . Then there is a family  $\{F_\gamma\}$  of  $2^{\mathbf{M}}$  subsets of  $X$  such that

- (1) For  $\gamma \neq \gamma'$ ,  $|F_\gamma \setminus F_{\gamma'}| = \mathbf{M}$ .
- (2) No  $F_\gamma$  admits of any continuous displacement of order  $\mathbf{M}$  onto itself or any other  $F_{\gamma'}$ .
- (3) For every  $\beta, \gamma$ ,  $|F_\gamma \cap K_\beta| = \mathbf{M}$ , and  $|(X \setminus F_\gamma) \cap K_\beta| = \mathbf{M}$ .

PROOF. There exist only  $\mathbf{M}$   $G_{\mathbf{m}\delta}$ -sets in  $X$ , and a fixed subset of  $X$  admits at most  $\mathbf{M}$  continuous maps into  $X$ , and therefore at most  $\mathbf{M}$  continuous displacements of order  $\mathbf{M}$ . Let  $f_\beta$  be a continuous displacement of order  $\mathbf{M}$  whose domain is a  $G_{\mathbf{m}\delta}$ -set. The family  $\{f_\beta\}$  of all such mappings has cardinal at most  $\mathbf{M}$ . This family is non-empty (otherwise the theorem is trivial), so by counting a given displacement  $\mathbf{M}$  times if necessary, we may assume that  $|\{f_\beta\}| = \mathbf{M}$ .

Now we apply [2; Lemma 1], with  $X = N$ ,  $M = m$ , and  $\{f_\beta\}$ . We obtain a family  $\{F_\gamma\}$  of  $2^m$  subsets of  $X$  satisfying (1) and (3). Suppose (2) is false, and there is a continuous displacement of order  $M$ ,  $\varphi$ , from  $F_\gamma$  onto  $F_{\gamma'}$ . This  $\varphi$  can be extended (Theorem 2.4) to a continuous map  $\tilde{\varphi}$  of a  $G_m$ -set  $G_\gamma \supset F$  into  $X$ , so  $\tilde{\varphi} = f_\beta$  for some  $\beta$ . Hence, by [2; Lemma 1, (2.3)], for every pair  $\gamma, \gamma'$ ,  $f_\beta F_\gamma \setminus F_{\gamma'} \neq \emptyset$ , and so, since  $\varphi = f_\beta$  on  $F_\gamma$ ,  $\varphi F_\gamma \setminus F_{\gamma'} \neq \emptyset$ , i.e.,  $\varphi$  maps  $F_\gamma$  onto no member of  $\{F_\gamma\}$ .

**2.5. LEMMA.** Let  $P$  be a space in which every open set has cardinal at least  $M$ . If  $\varphi : P \rightarrow P$  is non-trivial, and is either locally topologically into  $P$  or continuous onto  $P$ , then  $\varphi$  is a displacement of order  $M$ .

**PROOF.** The proof is word for word the proof of [2; Lemma 2], with “ $N$ ” replaced by “ $M$ ”, and “continuous displacement” replaced by “continuous displacement of order  $M$ ”.

**2.6. THEOREM.** Let  $X$  be a locally compact Hausdorff space of weight  $m$ , complete in a uniformity generated by  $m$  pseudometrics, such that every open set in  $X$  has  $2^m$  points. Let  $K$  be the set of all compact subsets of  $X$  whose cardinal is  $2^m$ . Then the sets  $\{F_\gamma\}$  constructed in Theorem 2.4 are such that no  $\{F_\gamma\}$  can be mapped topologically into or continuously onto itself or any other  $F_{\gamma'}$ .

**PROOF.** Each open set in each  $F_\gamma$  will have  $2^m$  points. By Lemma 2.5 and (1), Theorem 2.4, any non-trivial  $\varphi$  satisfying either condition of the theorem is a continuous displacement of order  $M$ . But this contradicts (2), Theorem 2.4.

**2.7. EXAMPLE.** Theorem 2.6 enables us to construct many examples of rigid 0-dimensional spaces of arbitrary weight. For instance, let

$$X = \prod_{\alpha \in A} X_\alpha,$$

where  $|A| = m$ , and, for each  $\alpha$ ,  $X_\alpha$  is a discrete space of cardinal two. Then  $X$  has weight  $m$ ,  $X$  is compact, and hence complete in any uniformity, so  $X$  is complete in a uniformity generated by  $m$  pseudometrics. Further, every open set in  $X$  contains  $2^m$  points. Now, applying Theorem 2.6, we get a collection of  $2^{2^m}$  sets  $\{F_\alpha\}$ , each of weight  $m$  and dimension 0, such that  $F_\gamma$  is rigid for each  $\gamma$ , and the  $F_\gamma$  are topologically distinct.

**2.8. PROBLEM.** The rigid spaces constructed in the preceding example are proper dense subsets of a compact space, hence they

are not themselves compact. We have not been able to construct examples of compact, rigid 0-dimensional spaces of arbitrarily high weight; such spaces would be of interest in the study of Boolean rings (see, for example, [2; § 8.1]).

### III. Spaces whose Autohomeomorphism Groups are $S_\alpha$ or $S_\alpha + \Sigma C_2$ .

If  $\alpha$  is finite, the discrete space of cardinal  $\alpha$  has  $S_\alpha$  as its autohomeomorphism group. This is not the case for  $\alpha$  infinite, of course. In Example 3.1, however, we produce for each infinite  $\alpha$  a space having  $\alpha$  isolated points whose autohomeomorphism group is precisely  $S_\alpha$ . In Example 3.2, we find spaces whose autohomeomorphism group is the direct sum of  $S_\alpha$  and the sum of *continuously* many groups of order two; this group is then isomorphic to  $S + \Sigma C_2$  if we assume the continuum hypothesis. In this connection one should recall the remark following the proof of Theorem 1.3; it is conceivable that  $\aleph_1$  can be replaced by  $\mathfrak{c}$  throughout this paper.

In both 3.1 and 3.2, the spaces  $S_p$  which play a part in the construction can evidently be chosen to have arbitrarily high weight, hence the same is true for our examples.

**3.1. EXAMPLE.** Let  $P$  be a discrete space of cardinal  $\alpha$ , and let  $\beta P$  be its (0-dimensional) Čech-Stone compactification. With each  $p \in P$ , we associate a 0-dimensional space  $S_p$  such that

(1) for each  $p \in P$ ,  $S_p$  is rigid and dense-in-itself, and (2) if  $p$  and  $q$  are distinct elements of  $P$ , then no non-empty open subset of  $S_p$  is homeomorphic to an open subset of  $S_q$ .

Such a collection  $\{S_p\}$  can be constructed by using Example 2.7, as follows: with each  $p \in P$ , we associate a cardinal  $\alpha_p$  such that if  $p \neq q$ ,  $2^{\alpha_p} \neq 2^{\alpha_q}$ . Taking  $\alpha_p = \mathfrak{m}$  in 2.1, we obtain a rigid space which we can denote by  $S_p$  such that each open subset of  $S_p$  contains  $2^{\alpha_p}$  points. The collection  $\{S_p\}$ ,  $p \in P$  evidently satisfies (1) and (2).

Now let

$$X = \bigcup_{p \in P} S_p \cup \beta P.$$

We topologize  $X$  by prescribing a base for the open sets, consisting of

- (i) the sets  $\{p\}$ ,  $p \in P$ ,
- (ii) the open-and-closed sets in  $S_p$  for each  $p \in P$ ,
- (iii) the sets



$$U \cup \bigcup_{p \in U} S_p$$

where  $U$  is open-and-closed in  $P$ .

The space  $X$  so defined is evidently a 0-dimensional completely regular Hausdorff space. The topology on each  $S_p$  as a subspace of  $X$  is the same as its original topology.

Every mapping of  $X$  onto  $X$  which permutes a finite number of the (isolated) points of  $P$  and leaves all other points of  $X$  fixed, is clearly a homeomorphism. These are the only autohomeomorphisms of  $X$ . For if an autohomeomorphism  $\varphi$  leaves each  $p \in P$  pointwise fixed, then the points of  $\beta P$  are fixed, so

$$\bigcup_{p \in P} S_p$$

must be mapped topologically on itself. But from (1) and (2), this space is rigid, so  $\varphi$  is the identity map. On the other hand, if  $\varphi$  displaces an infinite subset  $D$  of  $P$ , then  $\varphi$  must move some point of  $\beta P \setminus P$  (since the closures of  $D$  and  $\varphi(D)$  in  $\beta P$  are non-empty and disjoint), hence there is a  $p \in P$  such that  $S_p \cap \varphi S_p = \emptyset$ . But  $\varphi S_p \cap \beta P = \emptyset$ , since no open set in  $S_p$  contains an isolated point. It follows that  $\varphi S_p \cap S_{p'} \neq \emptyset$  for some  $p \neq p'$ , contradicting (2).

**3.2. EXAMPLE.** For each  $\alpha$ , we construct a space  $T_\alpha$  such that  $A(T_\alpha)$  is precisely  $S_\alpha + \Sigma C_2$  (assuming the continuum hypothesis). Let  $M$  be a 0-dimensional subset of the real numbers such that  $A(M)$  is the direct sum of continuously many groups of order two [2; § 5, Example I], and let  $X$  be the space constructed in Example 3.1, so that  $A(X) = S_\alpha$ . Let  $T_\alpha = X \cup M$ . If  $\varphi$  is any autohomeomorphism of  $T$ , then  $x \in M$  if and only if  $\varphi(x) \in M$ , since  $x \in M$  if and only if the least cardinal of a base at  $x$  is  $\aleph_0$ . It follows that  $A(T_\alpha) = A(X) + A(M) = S_\alpha + \Sigma C_2$ .

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